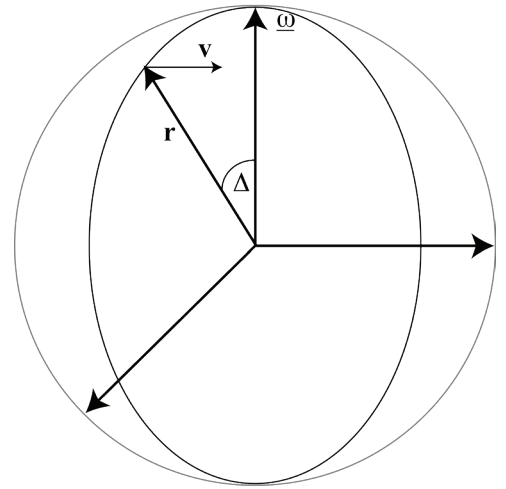# Plate Motions on a Sphere

Given: $\underline{\omega}$ − angular velocity vector ($rad/_s$)

$\underline{r}$: position on Earth's surface (m)

Calculate: $\underline{v}$: velocity vector at $\underline{r}$ ($m/_s$)

Of course, the velocity of the plate must be tangent to the surface of Earth so the velocity is the cross product of the position vector and the angular velocity vector.

$$\underline{v} = \underline{\omega} \times \underline{r} \tag{1}$$

or

$$\hat{v} = \hat{x}(\omega_y z - \omega_z y) - \hat{y}(\omega_x z - \omega_z x) + \hat{z}(\omega_x y - \omega_y x) \tag{2}$$

where $\hat{x}, \hat{y}$ and $\hat{z}$ are unit vectors. The magnitude of the velocity is given by

$$\left| \underline{v} \right| = \left| \underline{\omega} \right| \left| \underline{r} \right| \sin(\Delta) \tag{3}$$

where Δ is the angle between the position vector and the angular velocity vector. It is given by the following formula involving a dot product:

$$\cos(\Delta) = \frac{w \bullet r}{\left| \underline{\omega} \right| \left| \underline{r} \right|} \tag{4}$$

The formulae above assume that the angular velocity vector and the position vector are provided in Cartesian coordinates. However, usually they are specified in terms of latitude and longitude - we must transform both vectors. The usual problem is to calculate the relative velocity between two plates somewhere along their common boundary. Table 2.1 (next page) lists the pole position and rates of rotation for relative motion between plate pairs. The Cartesian position of a point along the plate boundary is:
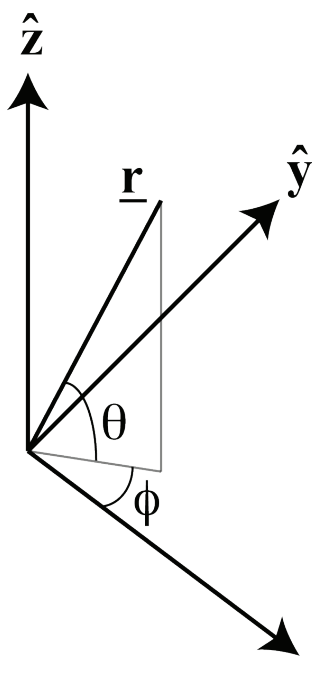
$$\begin{aligned} x &= a \cos\theta \cos\phi \\ y &= a \cos\theta \sin\phi \\ z &= a \sin\theta \end{aligned}$$

$$\begin{aligned} \omega &= \left| \underline{\omega} \right| \\ \omega_x &= \omega \cos\theta \cos\phi \\ \omega_y &= \omega \cos\theta \sin\phi \\ \omega_z &= \omega \sin\theta \end{aligned} \tag{5}$$

(6)

$\left| \underline{v} \right| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ *after computing* $\underline{v} = \underline{\omega} \times \underline{r}$ You should memorize the conversion from latitude and longitude to the Cartesian coordinate system although the conversion is quite simple (5). The **X** vector runs through 0° latitude and 0° longitude (i.e.

the Greenwich meridian); the **Y**-axis runs through a point at 0° latitude and 90° east longitude (recall the Nine-ty-East Ridge in the Indian Ocean); and the **Z** axis runs along the spin-axis of Earth to the north pole.

Similarly, the pole positions (of rotation) must be converted from geographic coordinates into the Cartesian system using the last set of equations in (5). There are two ways to compute the magnitude of the velocity. For example, use (1) to calculate the cross product of the rotation vector and the position vector. The magnitude of the velocity is then given by (7). A second approach is to calculate the angle Δ between the position vector and the angular velocity vector using (4) and then use that value in (3) to calculate the magnitude of the velocity. Turcotte and Schubert use this latter approach, but requires the use of spherical trigonometry to calculate Δ.

Velocity Azimuth

We know that the velocity vector is tangent to the sphere. Given the Cartesian velocity components from (2), we can compute the latitude (v_θ) and longitude (v_ϕ) components of velocity. Begin by taking the time derivative of (5):

$$\dot{x} = a\left(-\sin\theta\cos\phi\,\dot\theta - \cos\theta\sin\phi\,\dot\phi\right)$$
$$\dot{y} = a\left(-\sin\theta\sin\phi\,\dot\theta + \cos\theta\cos\phi\,\dot\phi\right) \tag{7}$$
$$\dot{z} = a\cos\theta\,\dot\theta$$

$$\dot\theta = \frac{\dot{z}}{a\cos\theta} \tag{8}$$

$$\dot{y} = a\left(-\sin\theta\sin\phi\,\dot\theta + \cos\theta\cos\phi\,\dot\phi\right)$$
$$\quad = a\left(-\sin\theta\sin\phi\,\frac{\dot{z}}{a\cos\theta} + \cos\theta\cos\phi\,\dot\phi\right)$$
$$\dot{y} + a\sin\theta\frac{\sin\phi}{a\cos\theta}\dot{z} = a\cos\theta\cos\phi\,\dot\phi$$
$$\dot\phi = \frac{\dot{y}}{a\cos\theta\cos\phi} + \frac{\sin\theta\sin\phi\,\dot{z}}{a\cos^2\theta\cos\phi}$$
$$\quad = \frac{1}{a\cos\theta\cos\phi}\left[\dot{y} + \frac{\sin\theta\sin\phi}{\cos\theta}\dot{z}\right]$$
$$\quad = \frac{1}{a\cos\theta\cos\phi}\left[\dot{y} + \tan\theta\sin\phi\,\dot{z}\right] \tag{9}$$

Or, if this is singular, derive using the x-velocity:

$$\dot{x} = a\left(-\sin\theta\cos\phi\,\frac{\dot{z}}{a\cos\theta} - \cos\theta\sin\phi\,\dot\phi\right)$$
$$\dot{x} + \frac{\sin\theta\cos\phi}{\cos\theta}\dot{z} = -a\cos\theta\sin\phi\,\dot\phi \tag{10}$$
$$\dot\phi = -\frac{\dot{x}}{a\cos\theta\sin\phi} - \frac{\sin\theta\cos\phi}{a\cos^2\theta\sin\phi}\dot{z} = -\frac{1}{a\cos\theta\sin\phi}\left[\dot{x} + \tan\theta\cos\phi\,\dot{z}\right]$$

In summary, to calculate the magnitude of the velocity:
      1) Transform lat, lon into $\underline{r}$=(x,y,z) unit vector using (5)
      2) Transform Euler pole lat, lon into $\underline{p}$=(x_p,y_p,z_p) unit vector using the bottom part of (5).
      3) $\cos\Delta = \underline{x}\cdot\underline{p}$
      4) $v = \omega a\sin\Delta$
An example:

Given the rotation pole between the Pacific and Nazca plates, calculate the spreading rate at 20°S, 113.5°W. Note that S latitudes are negative in our notation and that western latitudes are also negative. That is, the spherical coordinates that we will use are -20°, (360-113.5)° = -20°, 236.5°. MATLAB can make the calculations much less tedious, but the routines simply employ the expressions in (5).

[x y z]=sph2cart(deg2rad(246.5),deg2rad(-20),1)
x =
  -0.3747
y =
  -0.8618
z =
  -0.3420

[px py pz]=sph2cart(deg2rad(269.9),deg2rad(55.6),1)
px =
  -9.8605e-04
py =
  -0.5650
pz =
   0.8251

deg2rad is a Matlab function whose purpose is obvious. sph2cart converts spherical coordinates to Cartesian. It's important that you note in the documentation for Matlab that they switch the roles of θ and φ. The notes use θ as the latitude and φ as the longitude while Matlab uses the opposite convention.

First take the cross product $\underline{\omega}$ x $\underline{r}$:

C=cross([px py pz],[x y z])
C =
   0.9043   -0.3095   -0.2108

What's the magnitude of $\underline{v}$?

>> sqrt(C(1)^2+C(2)^2+C(3)^2)
ans =
0.9788

Now multiply by |$\underline{r}$| and |$\underline{\omega}$|:

>> 0.9788*6371*2.478e-8
ans =
1.5453e-04

This is the plate relative velocity in km/my.

Use (4) to calculate the cosine of Δ (note that both [x y z] and [px py pz] are already normalized since the radius of Earth and the amplitude of $\underline{\omega}$ were not included in the calculations above):

cosdelta=dot([Px Py Pz],[x y z])
cosdelta =
   0.2050
delta=acos(cosdelta)
delta =
   1.3643
deltadegree=rad2deg(delta)
deltadegree =
   78.1690

Now, include the radius of Earth (6371km) and the pole's rotation rate (2.478e-8 rad/my):

velocity=2.478e-8*6371*sin(delta)
velocity =
   1.5452e-04

Convert this rate to m/my:

velocitym=velocity*1000
velocitym =
   0.1545

and finally convert the rate to mm/my:

velocitymm=1000*velocitym
velocitymm =
   154.5196

The plate motion rate at r is 154.5196 mm/my.

Homework Problem:

The figure on the following page shows the San Andreas Fault in southern CA in the Carrizo Plain (courtesy, of course, of Google Earth). Using the Pacific-NA pole and the location of the fault (35°7'N, 119°39'W) to determine the slip vector along the fault. How closely does the prediction match the map?